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SS6

Surfaces with Non-Polyhedral Effective Cone

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The project started by a question of Keum and Lee [5]: do there exist minimal surfaces of general type with geometric genus zero which are not Mori dream spaces? Although recent work of Cascini, Catanese, Chen and Keum gives a positive answer in the setting of fake $\mathbf{Q}$-homology quadrics [2], it remains interesting to find effective criteria for detecting when an algebraic variety is not a Mori dream space, even if its geometric genus do not vanish.

Let $f\colon S \to B$ be a fibration from a smooth projective surface onto a smooth curve. We denote by $n(f)$ the weighted number of reducible fibres of f: a fibre with k distinct irreducible components contributes $k-1$. We proved

Theorem 1. *If the closed effective cone of S is polyhedral, then*

$$\rho(S) = n(f) + 2,$$

where $\rho(S)$ is the Picard number of S.

Since the closed effective cone of a Mori dream space is rational polyhedral, the theorem yields the following criterion: if $\rho(S) > n(f) + 2$, then $\overline{\mathrm{Eff}}(S)$ is not polyhedral; in particular, S is not a Mori dream space.

Applying this criterion we show that several surfaces of general type are not Mori dream surfaces. The list includes

- the Craighero–Gattazzo surface, a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 1$ [3];
- some Fermat surfaces;
- certain double covers of Hirzebruch surfaces, including some Horikawa surfaces [4];
- some Kodaira double fibred surfaces;
- some surfaces isogenous to a higher product;
- some product-quotient surfaces [1].

Combining the theorem with the Shioda–Tate formula [6], we also obtain:

Corollary 2. *If an elliptic surface with a section has polyhedral closed effective cone, then its Mordell–Weil group has rank zero.*

References

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